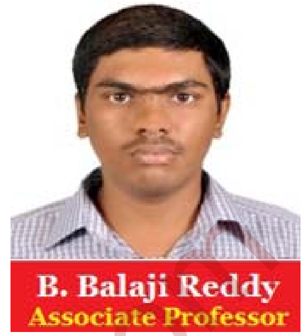
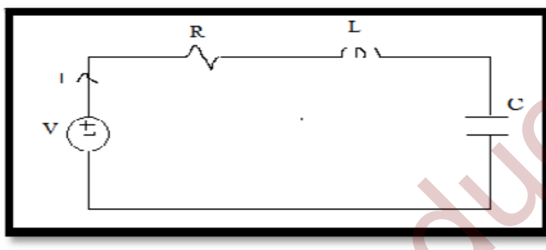
Electrical Resonance



Electrical resonance is the frequency response of an electrical network or circuit that's, the circuit operates at its natural frequency. At resonance condition the circuit exhibits unity power factor. So cos∅ =1 and ∅ =0. So the total supply voltage and the supply current are in phase and max power factor is transferred to the circuit under resonance. Electrical resonance can happen in any electrical circuit, when we have two similar but opposite natured elements like L and C. To undergo a good observable response we need a good quality in these energy storage elements. This is defined as Q-factor

$$\text{Q-factor} = 2\pi * \left[\frac{\max energy\ stored\ per\ cycle\ of\ supply}{energy\ disippiated\ per\ cycle\ of\ supply}\right]$$

Series resonance: at series resonance frequency, the total supply voltage and current are in phase.



So ∅ = 0.

Power factor = COS∅ =1 (UPF)

The net circuit behaves has purely resistive, so Z=R

Therefore X_{net} = 0

But Z = R + j (X_l -X_c)

If X_{net} =0 → (X_l -X_c) = 0

$$X_l = X_c \Rightarrow WL = \frac{1}{WC}$$

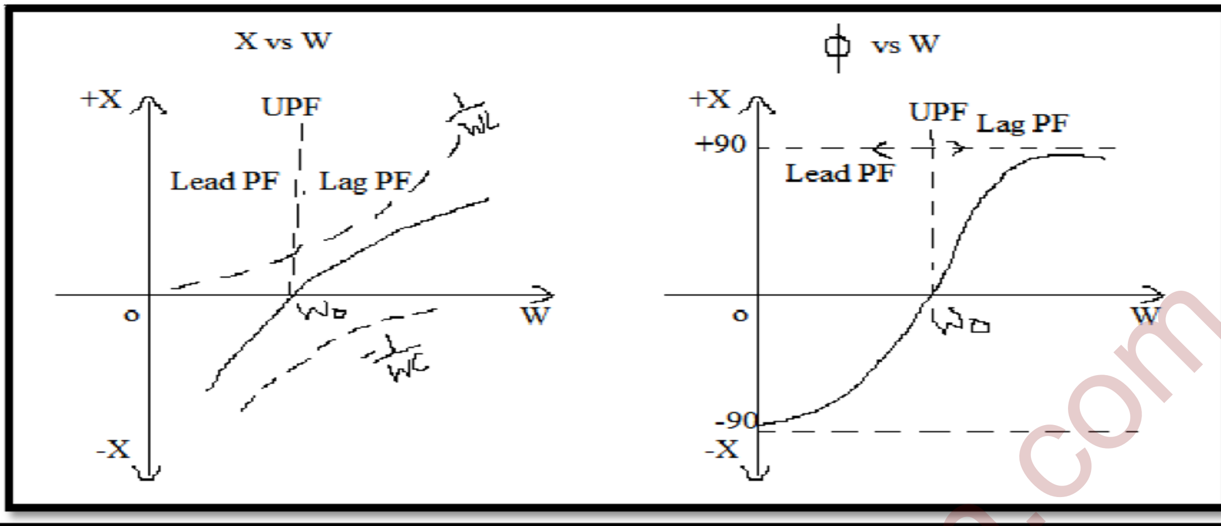
$$W^2\ LC = 1$$

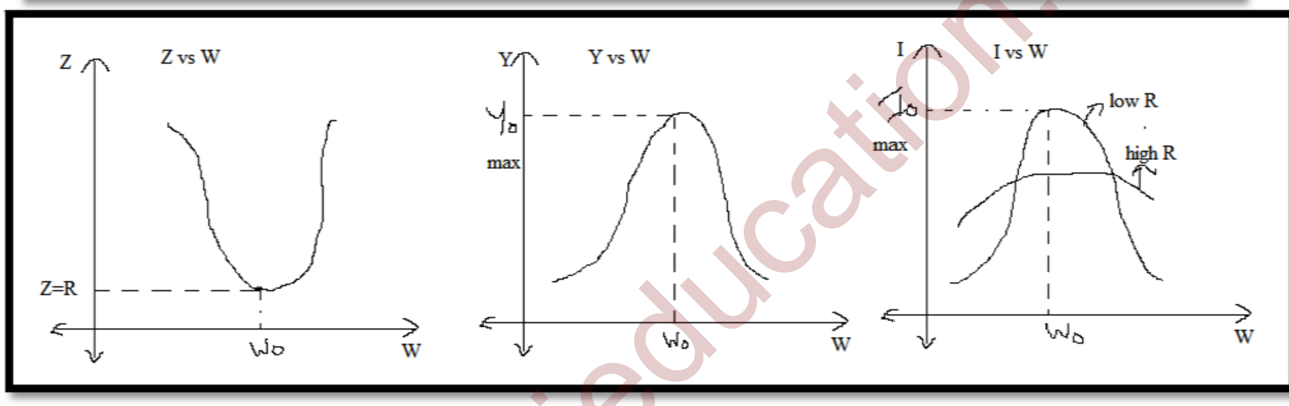
$$W^2 = \frac{1}{LC} \rightarrow W = \frac{1}{\sqrt{LC}}\ \text{rad/s}$$

$$(2\pi f)^2 = \frac{1}{\sqrt{LC}}\quad (\text{since } W = 2\pi f\ )$$

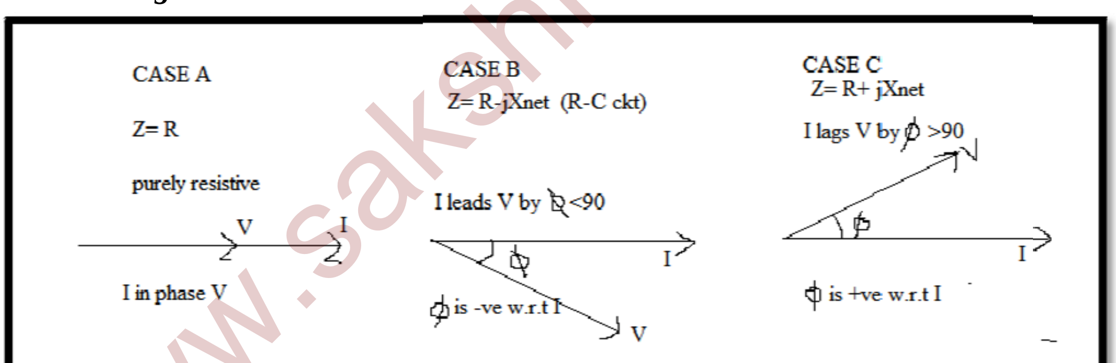
$$f = \frac{1}{2\pi\sqrt{LC}}\ \text{HZ}$$

Graphs:

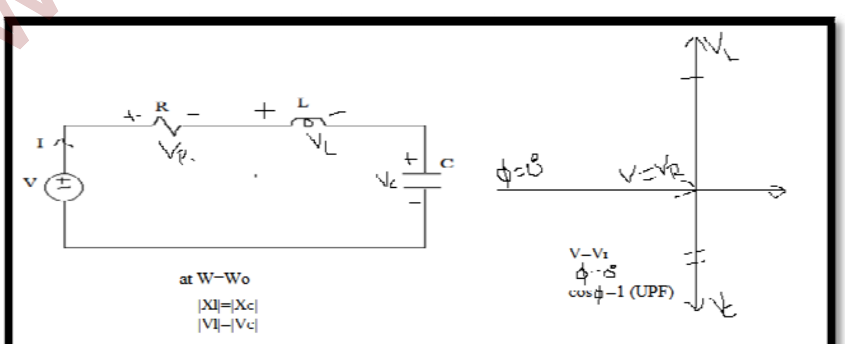




Phasor diagrams:



Complete phasor diagram:



Q-factor at resonance: $Q = \frac{WL}{R} = \frac{1}{WCR}$

$$\text{But } W = \frac{1}{\sqrt{LC}}$$

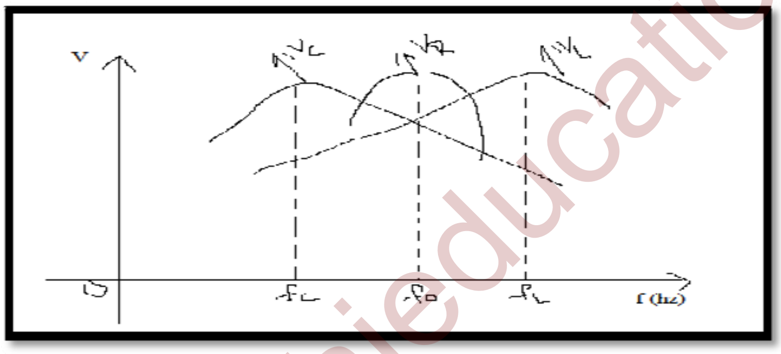
$$\text{Therefore } Q = \frac{1}{R}\sqrt{\frac{L}{C}}$$

Q-factor in series resonant circuit is also called as “voltage amplification factor”

At resonance condition, the net impedance is zero. So, the current is maximum that’s why series resonance circuit is called as “acceptor circuit”.

At resonance frequency the total supply voltage appears across resistor. So it is called as “voltage application circuit”.

Voltage across passive elements with change of frequency:



The frequency at which maximum voltage appears across the capacitor, $f_c = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{2L^2}}$ HZ

$$f_c = \frac{1}{2\pi\sqrt{LC}}\sqrt{1 - \frac{R^2C}{2L}} \text{ Hz} \quad \left(f_o = \frac{1}{2\pi\sqrt{LC}}\right)$$

$$f_c = f_o\sqrt{1 - \frac{R^2C}{2L}}$$

The frequency at which maximum voltage appears across the inductor is, $f_l = \frac{1}{2\pi\sqrt{LC - \frac{R^2c^2}{2}}}$

$$f_l = \frac{1}{2\pi\sqrt{LC}\sqrt{1 - \frac{R^2C}{2L}}}$$

$$f_l = \frac{f_o}{\sqrt{1 - \frac{R^2C}{2L}}}$$

From circuit, V = IZ

$|I| = \frac{|V|}{|Z|} = \frac{|V|}{\sqrt{R^2+(WL-\frac{1}{WC})^2}}$

At W=W_o → $I_o = \frac{|V|}{R}$ → maximum

So power delivered, $P_o = {I_o}^2$R watts → maximum

- V_R = IR → I_o R = $\frac{|V|}{R}$ * R → V_R = |V|
- V_l = +jX_lI = +$(W_oL)\frac{|V|}{R}$ = +jQ_o|V|→ V_l = +jQ_o|V|
- V_c = -jX_cI = -$\frac{j}{W_oc}\frac{|V|}{R}$ = -jQ_o|V|→V_c = -jQ_o|V|

Band width (BW):

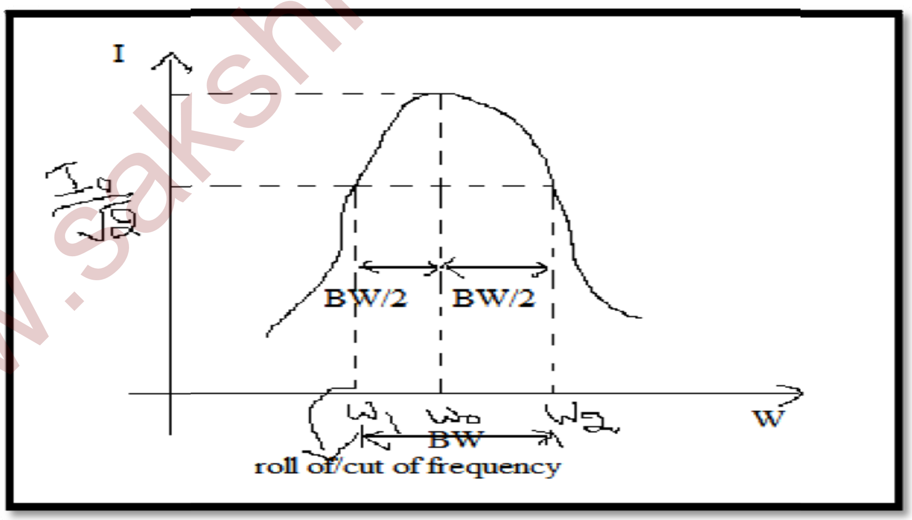
Band width represents the range of frequencies for which the power level in the signal is at least half of maximum power.

So, half power frequency is band width

$$P_o = {I_o}^2R$$

So half power, $\frac{P_o}{2} = \frac{{I_o}^2R}{2} = \frac{I_o}{\sqrt{2}}^2$ R = $(0.7070I_o)^2$ R

BW = W_2 - W_1 = $\frac{R}{L}$ rad



f_2- f_1 = $\frac{R}{2\pi L}$ Hz these are represents the geometric mean so that these can be represented as

$$W_o = \sqrt{W_2\, W_1}$$

$$f_o = \sqrt{f_2\, f_1}$$

W_o - W_1= $\frac{BW}{2}$ → W_o= W_1+ $\frac{BW}{2}$ → W_1 = W_o- $\frac{R}{L}$ rad/s

f_o- f_1 = $\frac{BW}{2}$ → f_o = f_1 + $\frac{BW}{2}$ → f_1 = f_o - $\frac{R}{2\pi L}$ Hz

W_2 - W_o= $\frac{BW}{2}$ → W_o= W_2- $\frac{BW}{2}$ → W_2 = W_o+ $\frac{R}{L}$ rad/s

f_2- f_o = $\frac{BW}{2}$ → f_o = f_2 - $\frac{BW}{2}$ → f_2 = f_o + $\frac{R}{2\pi L}$ Hz

Sensitivity:

Sensitivity is the ability to distinguish or discriminate between desired and undesired frequencies. It is also defined as the ratio of response frequency to bandwidth.

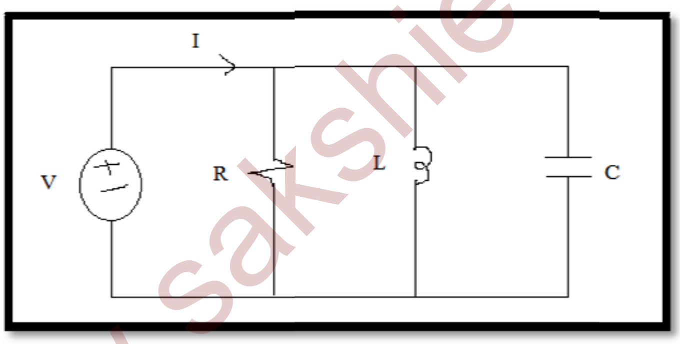
So, $S = \frac{f_o}{f_2 - f_o} = \frac{\frac{1}{2\pi\sqrt{LC}}}{\frac{R}{2\pi L}} = \frac{1}{R}\sqrt{\frac{L}{C}} = Q$ → S=Q

And $S \infty \frac{1}{|BW|}$

Parallel Resonance: At resonance condition, the supply voltage and current are in phase.

So Ø = 0.

Power factor = COSØ =1 (UPF)



The net circuit behaves has purely resistive, so net Admittance=conductance (Y = G)

Therefore B_{net} = 0 (net Susceptance)

- $Y_R = \frac{1}{Z_R} = \frac{1}{R}$
- $Y_L = \frac{1}{Z_L} = \frac{1}{jX_L} = \frac{-j}{X_L}$
- $Y_C = \frac{1}{Z_C} = -\frac{-1}{jX_C} = \frac{j}{X_C}$
- Y = Y_R+Y_L +Y_C → $Y = \frac{1}{R} + j\left(\frac{1}{X_C} - \frac{1}{X_L}\right) = \frac{1}{R} + j\left(WC - \frac{1}{WL}\right)$

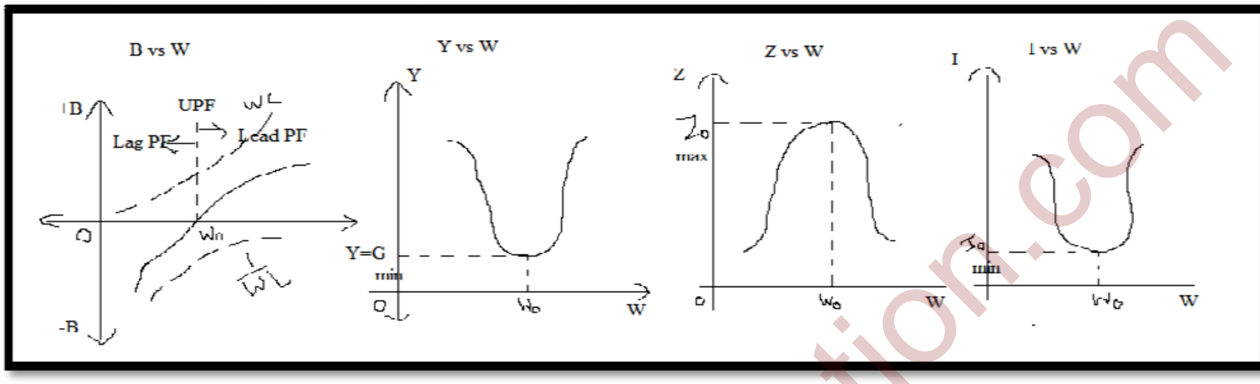
At resonance, W= W_o → B_{net} = 0

$$\frac{1}{X_C} = \frac{1}{X_L}$$

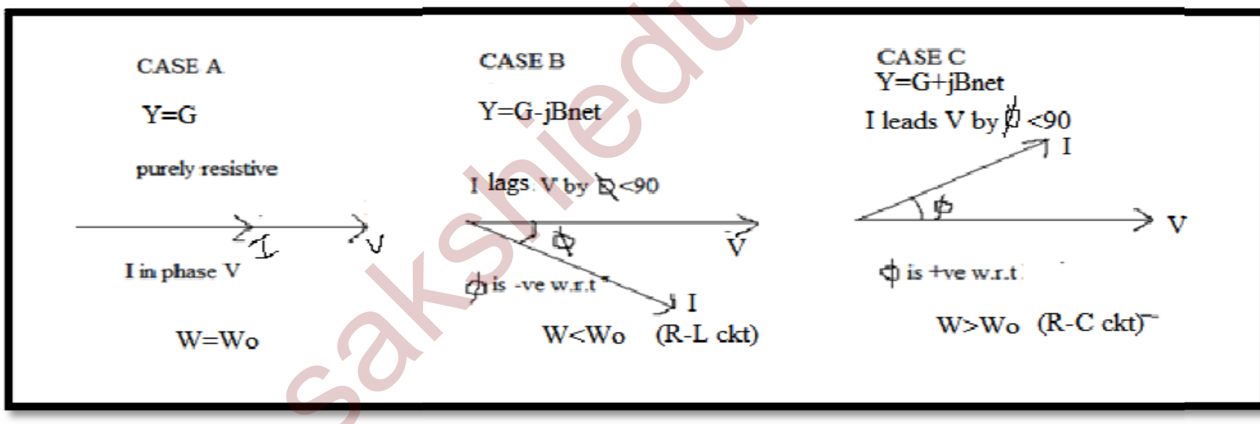
$$W_o C = \frac{1}{W_o L} \text{ rad/s}$$

$$f_o = \frac{1}{2\pi\sqrt{LC}} \text{ Hz}$$

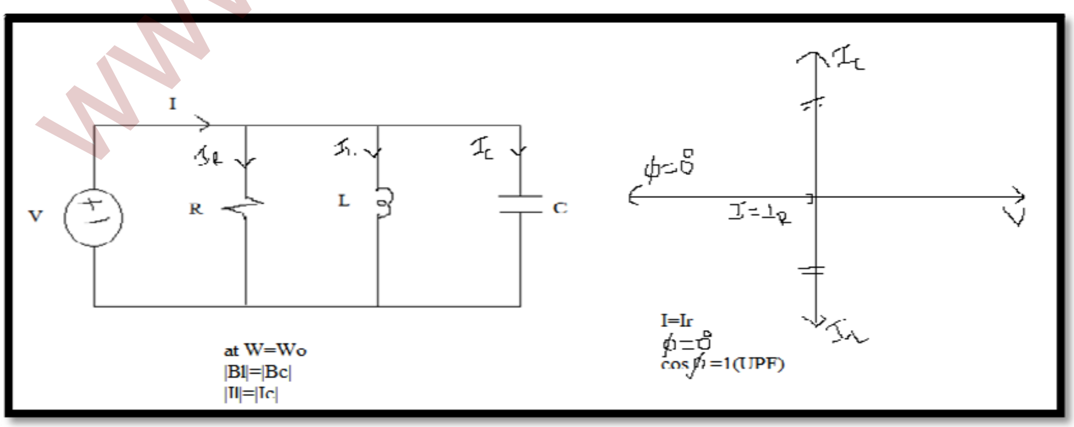
Graphs:



Phasor diagrams:



Complete phasor diagram:



Q-factor at resonance: Q = $\frac{R}{W_o L}$ = $W_o RC$

But W_o = $\frac{1}{\sqrt{LC}}$

Therefore Q = R$\sqrt{\frac{C}{L}}$

Q-factor in parallel circuit is called as "current amplification factor"

At parallel resonance condition, the net impedance maximum and current minimum so, it is called as "rejecter circuit".

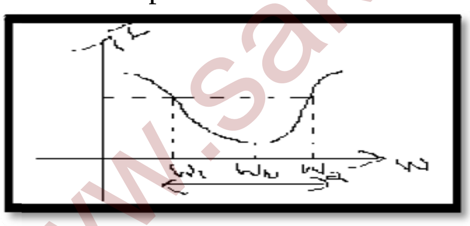
At parallel response frequency, the net current flows through resistor so; it is called as "current amplification circuit".

From circuit: V= IZ → |I| = $\frac{|V|}{|Z|}$

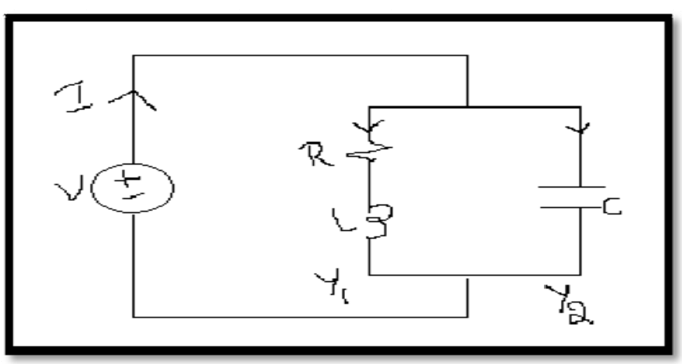
$$|V| = \frac{|I|}{|Y|} = \frac{|I|}{\sqrt{\frac{1}{R^2}+(WC-\frac{1}{WL})^2}}$$

W= W_o → |V| = $\frac{|I|}{\sqrt{\frac{1}{R^2}+0}}$ = |I| |R|

- I_R = $\frac{|V|}{|R|}$ → I_R = |I|
- I_l = $\frac{V}{+jX_l}$ = $\frac{|I|R}{+jW_oL}$ = -j[$\frac{R}{W_oL}$]|I|→ I_l = -jQ_o|I|
- I_c = $\frac{V}{-jX_c}$ = + [jW_oRC]|I| = -jQ_o|V|→I_c = +jQ_o|I|
- Parallel response circuit is also called as "band-stop-filter



Practical parallel response circuit (tank circuit → used in old radios):



$$Y_1 = \frac{1}{Z_1} = \frac{1}{R+jX_L} = \frac{R-jX_L}{R^2+{X_L}^2}$$

$$Y_2 = \frac{1}{Z_2} = \frac{1}{-jX_c} = \frac{j}{X_C}$$

$$\mathrm{Y} = Y_1 + Y_2 = \frac{R-jX_L}{R^2+{X_L}^2} + \frac{j}{X_C} = \left[\frac{R}{R^2+{X_L}^2}\right] + \mathrm{j}\left[\frac{1}{X_C} - \frac{X_L}{R^2+{X_L}^2}\right]$$

W= W_o → net susceptance = 0

$$\frac{1}{X_C} = \frac{X_L}{R^2+{X_L}^2}$$

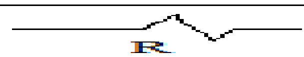
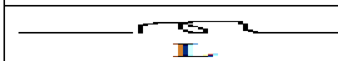
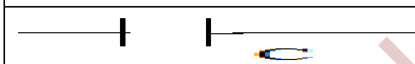
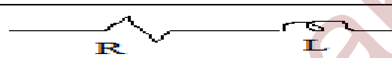
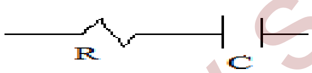
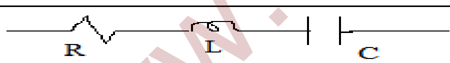
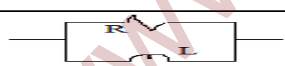
$${W_o}^2 L^2 = \frac{L}{C} - R^2$$

$${W_o}^2 = \frac{1}{LC} - \frac{R^2}{L^2}$$

$$W_o = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

$$f_o = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

Q-factor for different elements:

Element	Q-factor
	0
	∞
	∞
	$\frac{WL}{R}$
	$\frac{1}{WRC}$
	$\frac{1}{R}\sqrt{\frac{L}{C}}$
	$\frac{R}{WL}$
	WRC
	$\mathrm{R}\sqrt{\frac{C}{L}}$

Locus Diagrams & Resonance

Locus diagrams:

It is convenient to employ locus diagram for analyzing an electrical network having one variable element. It is either resistance or reactance. Since current in a circuit is with constant applied voltage, the current in the circuit is a function of impedance. So it is expressed as

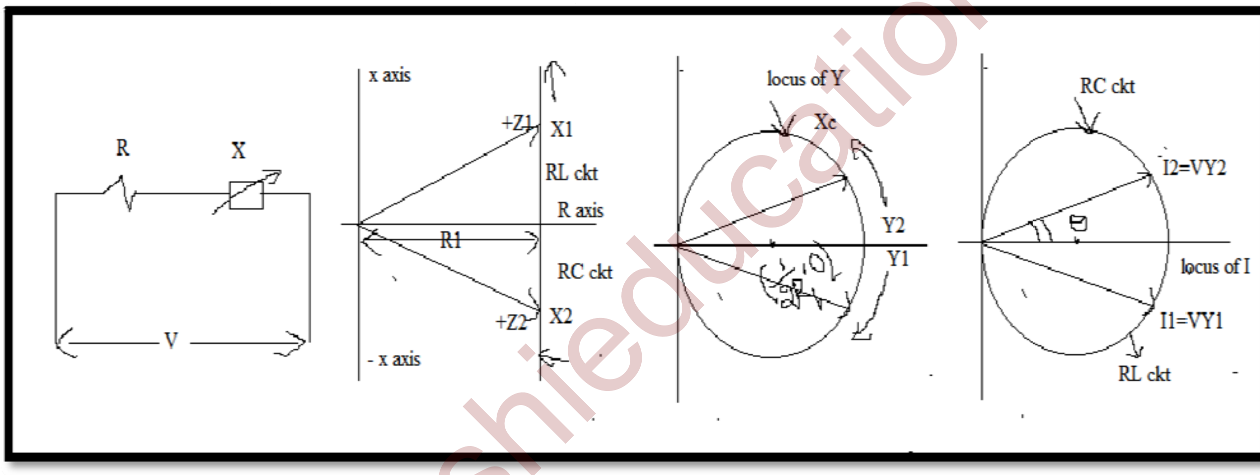
$$I = \frac{V}{Z} = VY$$

If V is constant, current is varies as that of admittance when one of the elements id changed. Hence the circuits are analyzed employing admittance locus diagram.

Locus diagram for series circuit:

Fixed resistance and variable reactance:

Consider a series circuit, having a fixed resistance R_1 and variable reactance either positive reactance X_l or negative reactanceX_c.



Since the resistance is constant, the variation of reactance results in impedance whose locus is straight line parallel to reactance axis and intersecting resistance axis at R_1as shown in fig, the locus on positive side of x axis correspond to RL circuit.

$I = \frac{V}{Z}$ and has a phase difference of$\tan^{-1}\frac{X}{R}$. The phasor sum ofV_R, V_X always equal to constant applied voltage V. The locus current is same as that of admittance. We will show that locus admittance is a circle.

From the series circuit, Z = $R_1 + jX$

$$Y = \frac{1}{Z} = G + j\,B$$

$$Z = R_1 + jX = \frac{1}{Y} = \frac{1}{G + j\,B}$$

$$Z = R_1 + jX = \frac{G - j\,B}{(G + j\,B)(G + j\,B)}$$

$$= \frac{G}{G^2 + \boldsymbol{B}^2} - j\,\frac{B}{G^2 + \boldsymbol{B}^2}$$

Therefore equating real and imaginary parts we get,

$$R_1 = \frac{G}{G^2 + \boldsymbol{B}^2}$$

$$\mathrm{X} = \frac{-B}{G^2 + \boldsymbol{B}^2}$$

$$G^2 + \boldsymbol{B}^2 = \frac{G}{R_1} \rightarrow G^2 + \boldsymbol{B}^2 - \frac{G}{R_1} = 0$$

Solving the above equation by mathematical analysis we get,

Adding $(\frac{1}{2R_1})^2$ to get to $(a+b)^2$ form,

$$G^2 - \frac{G}{R_1} + (\frac{1}{2R_1})^2 + \boldsymbol{B}^2 = (\frac{1}{2R_1})^2$$

$$(G - \frac{1}{2R_1})^2 + \boldsymbol{B}^2 = (\frac{1}{2R_1})^2$$

Its looks like an equation of a circle represents the circle with center at $(\frac{1}{2R_1}, 0)$ and radius of $\frac{1}{2R_1}$ on admittance plane with G and B axes. Locus diagrams are shown in above fig, the locus above G axis is for RC circuit and locus below G axis for RL circuit.

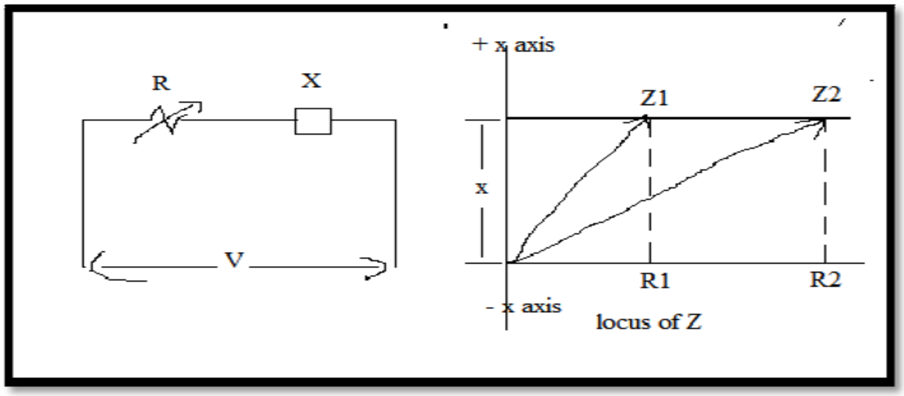
To draw current locus diagram:

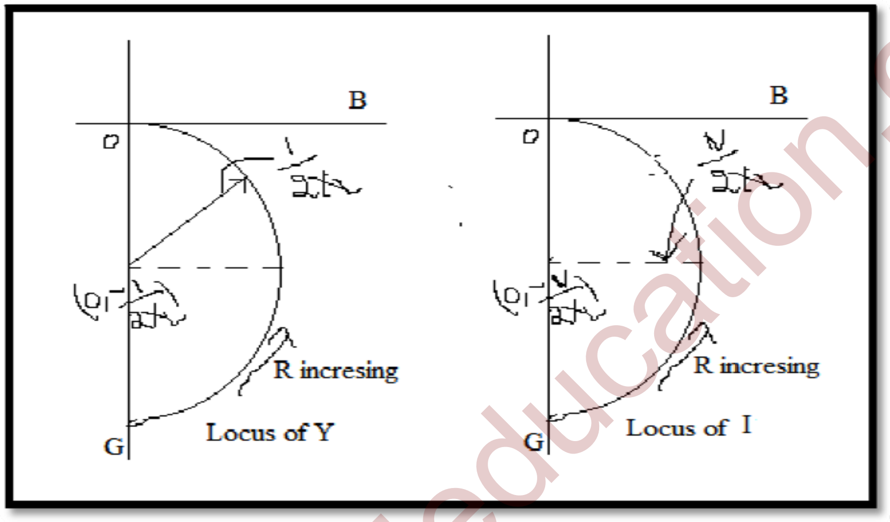
1) Draw the voltage phasor V as reference (x-axis)
2) Fix the center as $(\frac{V}{2R}, 0)$ and draw the circle radius with $\frac{V}{2\mathrm{R}}$ as radius where R is fixed radius.
3) For any value of x, draw a line through the origin making an angle $\tan^{-1}\frac{X}{R}$ and OA gives the current and θ gives phase angle.
4) VIcos θ, VI sin θ, cos θ gives active, reactive and power factor respectively.

Fixed reactance and variable resistance:

Consider a series circuit with a fixed reactance is positive or negative and a variable reactance R. The impedance of the circuit is

$$\mathrm{Z} = \mathrm{R} + \mathrm{j}\, X_l$$





If R is varied keeping X_l constant, then the locus of Z is a straight line parallel to R-axis and cutting X-axis at fixed reactance X_l. The locus of Y is a circle located at (0, $-\frac{1}{2X_l}$) and radius $\frac{1}{2X_l}$.

From the series circuit, Z = $R + jX_l$

$$Y = \frac{1}{Z} = G + j B$$

$$Z = R + jX_l = \frac{1}{Y} = \frac{1}{G + j B}$$

$$Z = R + jX_l = \frac{G - j B}{(G + j B)(G + j B)}$$

$$= \frac{G}{G^2 + B^2} - j \frac{B}{G^2 + B^2}$$

Therefore equating real and imaginary parts we get,

$$R = \frac{G}{G^2 + B^2}$$

$$X_l = \frac{-B}{G^2 + B^2}$$

$$G^2 + \boldsymbol{B}^2 = \frac{-B}{X_l} \rightarrow G^2 + \boldsymbol{B}^2 + \frac{B}{X_l} = 0$$

Adding both sides $(\frac{1}{2X_l})^2$ and we get,

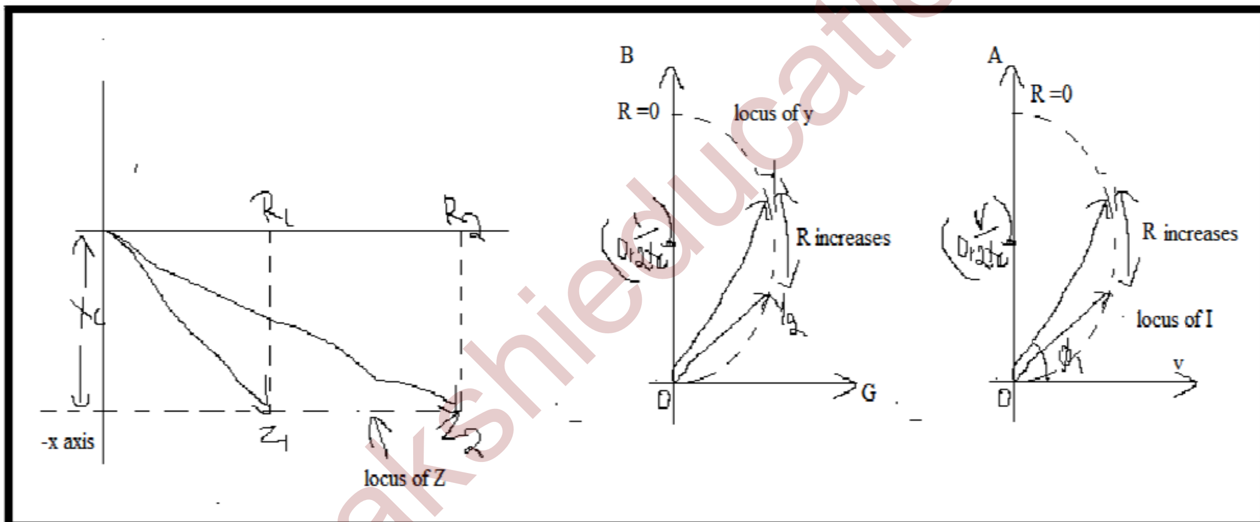
$$G^2 + \boldsymbol{B}^2 + \frac{B}{X_l} + (\frac{1}{2X_l})^2 = (\frac{1}{2X_l})^2$$

$$G^2 + (B + \frac{1}{2X_l})^2 = (\frac{1}{2X_l})^2$$

Which represents a circle in G-B plane with center at (0, -$\frac{1}{2X_l}$) and radius$\frac{1}{2X_l}$.

The locus of current is same as that of admittance obtained by multiplying with voltage V. it is a circle with center (0, -$\frac{V}{2X_l}$) and radius$\frac{V}{2X_l}$. (RL circuit)

If the reactance is capacitive, then the locus of Z, Y and current will be as shown fig. (RC circuit)



The current locus is as shown and it is a polar plot representing the variation of current for different values of R. the locus of current is a circle with (0,$\frac{V}{2X_c}$) and radius$\frac{V}{2X_c}$. To determine the current for any particular value of R, determine the phase angle = $\tan^{-1}\frac{X_c}{R}$ and draw a line OA cutting the locus of current at A. then OA represents the current and $\emptyset$ its phase angle.

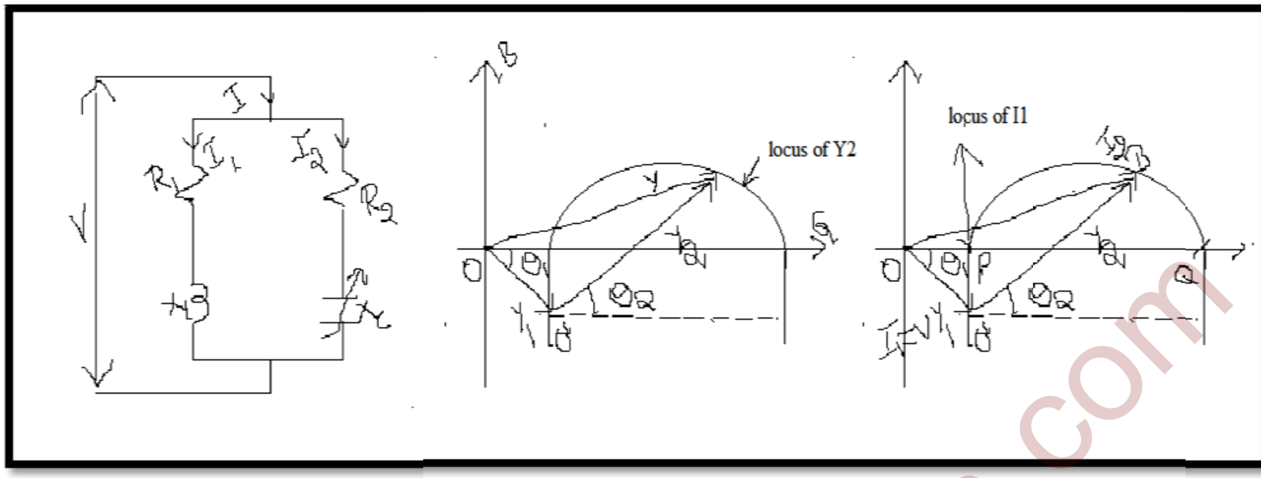
Locus Diagram for Parallel Circuits:

The method of solving networks using locus diagrams can be extended to parallel circuits in which one of the branches contain a variable element.

Fixed resistance and variable reactance:

Let us consider a two branch parallel circuit as shown in fig. the locus of admittance and current are as shown in fig.

For the branch 1, there is no variable element Z_1 = R_1 + jX_l which is constant.



So Z_1 = $|Z_1| < \theta_1$ and Y = $\frac{1}{Z_1 < \theta_1}$ = $Y_1 < -\theta_1$

The admittance Y_1 is first drawn to (OO^1) as shown in fig.

The impedance of second branch is Z_2 = R_2 - jX_c, in which X_c is a variable. The locus of Y_2 of branch 2 is a circle with center at ($\frac{1}{2R_2}$, 0) and radius of $\frac{1}{2R_2}$ (from series circuit concept). We can draw the locus of Y_2 from the tip of Y_1 so that any point on the semi circle, if joined to the origin O gives the total admittance, Y = Y_1+ Y_2. The semi circle locus represents the locus of total admittance Y with respect to origin with respect toO^1 give the locus of Y_2. The corresponding locus is obtained by multiplying with V and is shown in fig.

OO^{11} represents the current in Z_1 = I_1

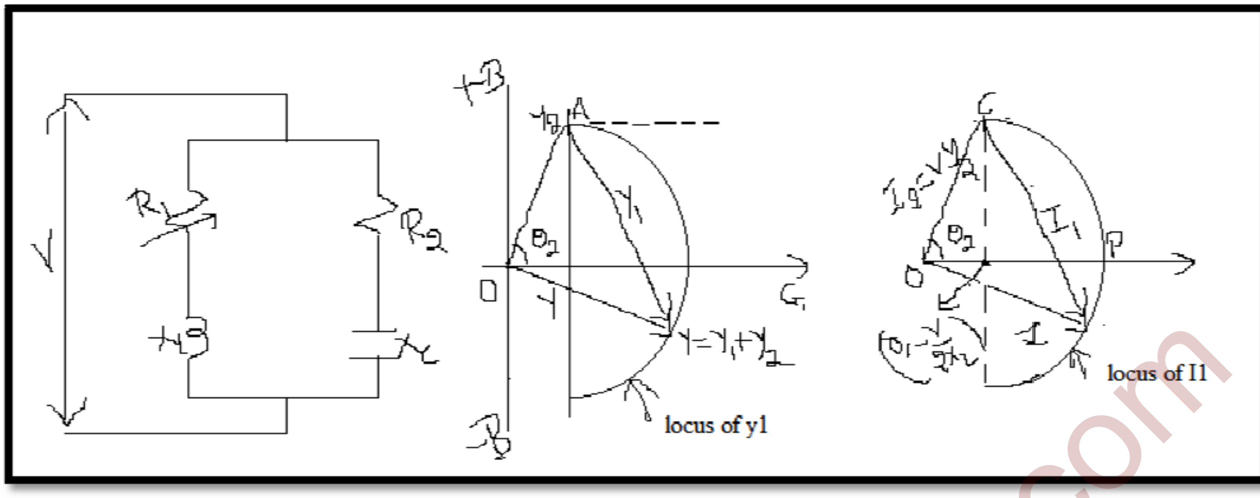
OO^{11} B current in Z_2 for a given value of X_c = I_2

OB total current, I = I_1 + I_2

From the locus diagrams, we can locate the points of resonance. At resonance, we know the current and voltage are in phase P, Q is unity power factor points where the current locus cuts at the voltage axis. For the given parallel circuit, we have two different values of X_c at which we have a resonance. If the radius of a circle $\frac{V}{2R_2}$ is such that the locus does not intersect the voltage phasor, then there will not be any value of X_c which makes the circuit to resonant.

Variable resistance and fixed reactance:

Let us consider a parallel circuit, as shown in fig. the admittance of locus also shown in below fig.



The impedance of branch 2 is fixed, Z_2 = R_2 - jX_c which is constant.

So $Z_2 = |Z_2| < -\theta_2$ and $Y_2 = \frac{1}{Z_2 < \theta_2} = Y_2 < \theta_2$

The admittance Y_2 is first drawn to (OA) as shown in fig.

The impedance of first branch is $Z_1 = R_1 + jX_L$, in which R_1 is a variable. The locus of Y_1 of branch 1 is a circle with center at (0, -$\frac{1}{2X_l}$) and radius $\frac{1}{2X_l}$ (from series circuit concept). The locus of Y_1 is drawn from the tip of Y_2 as shown in fig. The total current I = VY and the locus of I is as shown in fig, it is obtained from the admittance locus by multiplying with V. The semi circle represents the locus of I_1 with respect to C and with respect to origin O it gives the total current I. The point at which the current locus cuts the voltage phasor gives the resonant point P.